

In a nutshell: Approximating values using least-squares best-fitting polynomials

Least-squares best-fitting linear polynomials

Given equally-spaced points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots$, where the x values are equally spaced by a value h , we can find the linear polynomial that approximates the value at $x_k + \delta h$ as follows:

1. Create the Vandermonde matrix $V = \begin{pmatrix} -n & 1 \\ 1-n & 1 \\ \vdots & \vdots \\ -2 & 1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$ and the vector $\mathbf{y} = \begin{pmatrix} y_{k-n} \\ y_{k-n+1} \\ \vdots \\ y_{k-2} \\ y_{k-1} \\ y_k \end{pmatrix}$.
2. Find $(V^T V)^{-1} V^T$ which can be done most succinctly by calculating $\det(V^T V)$, which must be an integer, and then $\det(V^T V)(V^T V)^{-1} V^T$ must be an $(n+1) \times 2$ integer matrix.
3. Thus, we note that $(V^T V)^{-1} V^T \mathbf{y}$ defines a_1 and a_0 as linear combinations of the y -values.

For example, if we want to find the coefficients a_1 and a_0 for eight points, we have

$$\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \frac{1}{\det(V^T V)} \left(\det(V^T V) (V^T V)^{-1} V^T \right) \mathbf{y} = \frac{1}{336} \begin{pmatrix} -28 & -20 & -12 & -4 & 4 & 12 & 20 & 28 \\ -56 & -28 & 0 & 28 & 56 & 84 & 112 & 140 \end{pmatrix} \mathbf{y}$$

and thus, our approximation of $x_k + \delta h$ would be $a_1 \delta + a_0$.

Least-squares best-fitting quadratic polynomials

Given equally-spaced points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots$, where the x values are equally spaced by a value h , we can find the linear polynomial that approximates the value at $x_k + \delta h$ as follows:

1. Create the Vandermonde matrix $V = \begin{pmatrix} n^2 & -n & 1 \\ (1-n)^2 & 1-n & 1 \\ \vdots & \vdots & \vdots \\ 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and the vector $\mathbf{y} = \begin{pmatrix} y_{k-n} \\ y_{k-n+1} \\ \vdots \\ y_{k-2} \\ y_{k-1} \\ y_k \end{pmatrix}$.
2. Find $(V^T V)^{-1} V^T$ which can be done most succinctly by calculating $\det(V^T V)$, which must be an integer, and then $\det(V^T V)(V^T V)^{-1} V^T$ must be an $(n+1) \times 3$ integer matrix.
3. Thus, we note that $(V^T V)^{-1} V^T \mathbf{y}$ defines a_2, a_1 and a_0 as linear combinations of the y -values.

For example, if we want to find the coefficients a_2, a_1 and a_0 for eight points, we have

$$\begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \left((V^T V)^{-1} V^T \right) \mathbf{y} = \frac{1}{56448} \begin{pmatrix} 2352 & 336 & -1008 & -1680 & -1680 & -1008 & 336 & 3252 \\ 11760 & -1008 & -9072 & -12432 & -11088 & -5040 & 5712 & 21168 \\ 7056 & -2352 & -7056 & -7056 & -2352 & 7056 & 21168 & 39984 \end{pmatrix} \mathbf{y}$$

and thus, our approximation of $x_k + \delta h$ would be $(a_2 \delta + a_1) \delta + a_0$.